The Curvature of the Relativistic Rotating Disk

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1. Introduction

The case of a rigid disk rotating at relativistic speeds raises a number of interesting paradoxes and has long been plagued with misunderstandings. According to special relativity, measuring rods laid out along the rim of the disk will be Lorentz contracted according to the usual formula, but those laid out along the radius will not, as these are perpendicular to the (instantaneous) direction of motion. Thus, the ratio of the circumference to the diameter of the disk will no longer be π . This paradox was first introduced in 1909 by Paul Ehrenfest, and is referred to as Ehrenfest's paradox [1].

The Ehrenfest paradox was known to Einstein, and he actually used the case of a rotating disk in his seminal 1916 paper to introduce the necessity for non-Euclidian geometry in general relativity (GR) [2]. However, he never published a paper directly addressing the rotating disk. Other physicists, such as Strauss [3], argued that if the measuring rods were contracted, then so were the distances they were measuring, so the ratio C/D would still be π .

Another closely related difficulty which helps shed some light on possible resolution might be called the non-transitivity of synchronicity along the rim. For simplicity, let us represent the infinite number of instantaneously inertial frames on the edge of the disk by just four, K_1, K_2, K_3 and K_4 (see diagram):

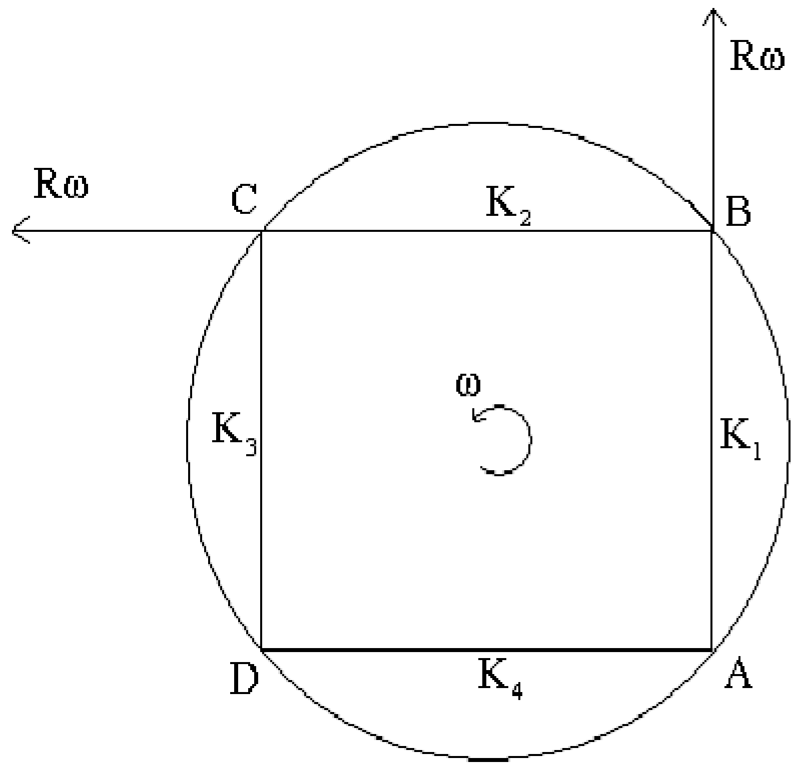

Figure 1

Letters A, B, C and D represent events (such as light flashes) that occur on the rim of the disk. Furthermore, let us assume that A and B are simultaneous in K_1, B and C are simultaneous in K_2 C and D are simultaneous in K_4. If A and B are simultaneous in K_1, then when we transform to K_2, A will be before B. Likewise, in K_3 B will occur before C and in K_4 C will occur before D. Lastly transforming back into K_1, we see that A occurs before D. Thus, by moving from frame to frame around the disk we see that A occurs before B, which occurs before C, which occurs before D, which occurs before A. Obviously, a discontinuity in time has occurred, and there is a flaw in this analysis.

A still more fundamental and thus disturbing paradox has been raised by Selleri in an interesting paper [4]. Using very elementary considerations, it can be shown that, based only on the assumption that the circumference is a well-defined geometric entity, the ratio of the co-rotating to counter-rotating speeds of light is given by

$$\Omega = \frac{c_-}{c_+} = \frac{1+\beta}{1-\beta}, \tag{1.1}$$

where ω is the (assumed constant) angular speed, R is the radius of the disk, $\beta = \frac{R\omega}{c}$, and c_+ and c_- are the velocities of light in the co- and counter-rotating direction, respectively. The paradoxical consequences can be seen by letting $R \to \infty$ and $\omega \to 0$ in the equation (1.1). In such a limit, we see that the centrifugal acceleration $a = \omega^2 R \to 0$ while allowing one to keep the tangential velocity $v = \omega R$ constant. Thus, the rim of the disk approaches an inertial frame in the limit, which is moving at a speed $v = \omega R$ with respect to a non-rotating observer at the origin ($\rho = 0$). This creates a discontinuity at $\omega = 0$, since we know that in an inertial frame, the speed of light is the same in the co- and counter-rotating directions, i.e. $\Omega = 1$.

There are experimental confirmations of apparent anisotropies of the speed of light on the rotating disk. If a laser beam is split, sent along the edge of a rapidly rotating disk (by means of a cylindrical mirror or similar device) and recombined at an interferometer located near the beam splitter, a phase shift is noted. Specifically, the time delay between the arrival of the counter-rotating beam and the co-rotating one is given [5] by

$$\Delta t' = \frac{2\pi\beta^2}{\omega\sqrt{1-\beta^2}}. \tag{1.2}$$

This effect was first noted in 1913 by Sagnac, and is called the Sagnac effect.

At first glance, it may appear as though, insofar as the rotating disk is an accelerated frame is ever there was one, general relativity must be used in calculations. Although the tensor calculus is used extensively in this paper, the physics is all essentially special relativity (SR). In the words of Einstein, "Kinematic equivalence of two coordinate systems is actually not restricted to the case when systems K and K' make rectilinear uniform motions. From the kinematic standpoint, this equivalence is fairly well satisfied, for instance, if one system uniformly rotates with respect to the other." [6] In other words, GR is unnecessary when considering purely kinematic effects.

2. Early Perspectives

The Sagnac Effect, having been measured repeatedly in the laboratory, even in one case using the earth as the rotating disk, is perhaps the most pressing difficulty raised in the introduction. Not surprisingly, there is extensive literature on the topic. Malykin [6] reviews the existing literature on the topic exhaustively, and the matter may be considered resolved. My intention in this paper is not to explain the Sagnac Effect (that has been done), but rather to clarify or explain away apparent paradoxes concerning the rotating disk in SR.

The most far-fetched explanations of these paradoxes claim that SR does not work on a rotating disk, i.e. that the speed of light is locally anisotropic on the disk, and that Galilean velocity addition is valid in the rotational sense. At least one author [7] has even gone as far as to re-derive kinematics on the disk, using the Galilean velocity addition as a postulate, and following the method used by Einstein in deriving SR. It may seem surprising that such claims appear in the literature, but they are surprisingly prevalent.

That such "resolutions" are physically unacceptable is obvious. The goal of physics is to use one set of axioms to describe all frames. Having separate theories for rotational and rectilinear motion is epistemologically untenable. Moreover, the success of relativity theory permeates almost all of physics and its replacement by such an ad hoc theory should certainly be a last resort.

As mentioned earlier Einstein himself did consider the problem. He, as well as a number of other prominent physicists, claimed that the circumference, $C \geq 2\pi R$. This is based on the consideration that the circumference in the non-rotating frame, C_0 and the circumference in the rotating frame C are geometrically equivalent. However, the measuring rods on the rotating platform will be Lorentz contracted according to the usual formula, so that the circumference will be measured to be $C_0 = \dfrac{2\pi R}{\left(1-\dfrac{v^2}{c^2}\right)^{\frac{1}{2}}}$

Einstein did not consider it to be a significant problem, however. He introduced a hypothesis that stated that the disk would bend and curve in such a way as to accommodate the Lorentz contraction. Aside from the rather ad hoc nature of such a hypothesis, it is evident that such a curvature would violate the assumed symmetries in the problem. If, for instance, the disk were to warp in the +z direction, that would introduce a skew in space and would violate the spatial parity of the inertial frame.

3. Circumference as Measured by Different Observers

Let us consider what an observer, at rest with respect to the rim of the disk, measures the circumference to be. Consider a grid of rods covering the disk with uniform clocks at every point, synchronized in the standard way. To an observer at the origin O, a point on the rim A moves with a velocity ωR and thus he measures a length of an infinitesimal portion of the rim to be $Rd\theta\sqrt{1-\frac{\omega^2 R^2}{c^2}}$. This contraction is illusory, however, in the sense that it is observer-dependent; the observer on the rim will not observe a contraction in his immediate neighborhood. He or she agrees that his or her instantaneously commoving frame has a speed ωR with respect to O, but will measure the length of a small portion of the rim to be simply $Rd\theta$.

Let us suppose such an observer is looking at a different point on the rim B, and that an angle of θ is subtended between A and B. To an observer at O, point A has a velocity $\mathbf{v}_A = \mathbf{i}\omega R$ and point B has velocity $\mathbf{v}_A = \mathbf{i}\omega R\cos\theta + \mathbf{j}\omega R\sin\theta$. Transforming to the reference frame of A (indicated by primes), we find that an observer at A measures the following velocity components:

$$u'_x = \frac{v(\cos\theta - 1)}{1-\frac{v^2}{c^2}\cos\theta} \qquad \text{and} \qquad u'_y = \frac{v\sin\theta\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v^2}{c^2}\cos\theta}, \tag{2.3}$$

where we have let v = ω R. So,

$$v' = \sqrt{u_x^2 + u_y^2} = \frac{v\sqrt{2 - 2\cos\theta - \frac{v^2}{c^2}\sin^2\theta}}{1 - \frac{v^2}{c^2}\cos\theta} = \frac{2v\sin\frac{\theta}{2}\sqrt{1 - \frac{v^2}{c^2}\cos^2\frac{\theta}{2}}}{1 - \frac{v^2}{c^2}\cos\theta} \tag{2.4}$$

and

$$\gamma' = \left(1 - \frac{v'^2}{c^2}\right)^{\frac{1}{2}} = \frac{4v^2\sin^2\frac{\theta}{2}\left(1 - \frac{v^2}{c^2}\cos^2\frac{\theta}{2}\right)}{c^2 - v^2\cos\theta} = \frac{1 - \frac{v^2}{c^2}\cos\theta}{1 - \frac{v^2}{c^2}}. \tag{2.5}$$

$$C' = \oint_C dl' = \sqrt{1 - \frac{v^2}{c^2}}\int_0^{2\pi}\frac{1}{\gamma'}Rd\theta = R\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}\int_0^{2\pi}\frac{d\theta}{1 - \frac{v^2}{c^2}\cos\theta} = 2\pi R\frac{1 - \frac{v^2}{c^2}}{\sqrt{1 + \frac{v^2}{c^2}}} \tag{2.6}$$

Compare this to the circumference measured by a non-rotating observer at O,

$$C = \int_0^{2\pi}\frac{1}{\gamma}Rd\theta = R\sqrt{1 - \frac{v^2}{c^2}}\int_0^{2\pi}d\theta = 2\pi R\sqrt{1 - \frac{v^2}{c^2}}\ . \tag{2.7}$$

Thus we see that $C' \le C \le 2\pi R$. Observers at different radii will in general not agree on the length of the circumference. Evidently, the various conception of the circumference are not "geometrically identical," as was the assumption of Lorentz, Ehrenfest, and others.

4. Curvature of the Disk

Parity violation notwithstanding, given the apparently non-Euclidean nature of the surface of a rapidly rotating disk, it is natural to inquire as to whether it possesses a Gaussian curvature. The problem naturally lends itself to the use of cylindrical coordinates. We will start with the metric in flat space-time,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad (4.1)$$

where $x^0 = ct$, $x^1 = \rho$, $x^2 = \phi$ and $x^3 = z$.

For an observer standing at the origin, the coordinate transformations from an inertial frame into the rest frame of a disk rotating at a constant angular velocity, ω, are given by,

$$t' = t$$

$$\rho' = \rho$$

$$\phi' = \phi + \omega t$$

$$z' = z \qquad (4.2)$$

By using the invariance of the interval,

$$c^2 d\tau^2 = c^2dt^2 - d\rho^2 - \rho^2 d\phi^2 - dz^2,$$

where τ is proper time of a particle at rest in the system, and

$$d\phi'^2 = d\phi^2 + 2\omega\, d\phi\, dt + \omega 2dt^2,$$

we find that

$$c^2\, d\tau^2 = (c^2 - \rho^2\omega^2)dt^2 - 2\omega\,\rho^2 d\phi\, dt - d\rho^2 - \rho^2 d\phi^2 - dz^2, \tag{4.3}$$

from which can be extracted the metric in the primed coordinate system,

$$g_{\mu\nu} = \begin{bmatrix} 1 - \frac{\rho^2\omega^2}{c^2} & 0 & -\frac{\omega\rho^2}{c} & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{\omega\rho^2}{c} & 0 & -\rho_2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{4.4}$$

If we assume that $d\rho = d\phi = dz = 0$, as is the case for a point at rest on the edge of the disk, we find that

$$d\tau = \sqrt{1 - \frac{\rho^2\omega^2}{c^2} dt^2} \tag{4.5}$$

This is ordinary time dilation. Taking the inverse of (4.4)

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & -\frac{\omega}{c} & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{\omega}{c} & 0 & \frac{\omega^2}{c^2} - \frac{1}{\rho^2} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{4.6}$$

From these the Christoffel symbols can be calculated according to the usual formula,

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}), \tag{4.7}$$

where ∂_a is shorthand for $\frac{\partial}{\partial x^a}$. They are

$$\Gamma^1_{22} = -\rho \qquad \Gamma^1_{00} = -\frac{\rho\omega^2}{c^2} \qquad \Gamma^2_{10} = \Gamma^2_{01} = \frac{\omega}{\rho c}$$

$$\Gamma^1_{20} = \Gamma^1_{02} = -\frac{\rho\omega}{c} \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{\rho}, \tag{4.8}$$

with all other Γ's zero. We can transform into rectangular coordinates ($x^0 \equiv ct$, $x^1 \equiv x$, $x^2 \equiv y$, $x^3 \equiv z$) to find,

$$\Gamma^1_{00} = -\omega^2 x \qquad \Gamma^2_{00} = -\omega^2 y$$

$$\Gamma^1_{20} = \Gamma^1_{02} = -\omega \qquad \Gamma^2_{10} = \Gamma^2_{01} = \omega \tag{4.9}$$

Substituting these into the general formula for geodesics on curved surfaces

$$\ddot{x}^a + \Gamma^a_{bc} x^b x^c = 0\,, \tag{4.10}$$

we obtain the geodesics for rectangular coordinates on the rotating disk:

$$\ddot{t} = 0 \tag{4.11}$$

$$\ddot{z} = 0 \tag{4.12}$$

$$\ddot{x} - \omega^2 x \dot{t}^2 - 2\omega \dot{y} \dot{t} = 0 \tag{4.13}$$

$$\ddot{y} - \omega^2 y \dot{t}^2 + 2\omega \dot{x} \dot{t} = 0\,, \tag{4.14}$$

where dots denote differentiation with respect to a parameter along the particle's path (typically proper time). Equation (4.11) implies that $\dot{t} = const$. Using this fact and multiplying by the mass m of the particle allows us to rewrite equations (4.12)-(4.14) as

$$m\frac{d^2 z}{dt^2} = 0 \tag{4.13}$$

$$m\frac{d^2 y}{dt^2} = m\omega^2 y - 2m\omega\frac{dx}{dt} \tag{4.14}$$

$$m\frac{d^2 x}{dt^2} = m\omega^2 x + 2m\omega\frac{dy}{dt} \tag{4.15}$$

or, in 3-vector notation,

$$m\frac{d^2\vec{r}}{dt^2} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{\omega}\times\frac{d\vec{r}}{dt} \tag{4.16}$$

which is the classical equation of motion for a free particle in a rotating reference system. Thus we see that an observer using t as his time "feels" Coriolis and centrifugal pseudoforces. Note that such psuedoforces (including gravity) are implicitly accounted for in the second derivatives of the metric.

From (4.7) or (4.8), one can find that the space-time is Riemann flat (i.e., R^{a}_{bcd} = 0), which is not surprising, since the original space was flat, and all we have really done so far is make a coordinate transformation. The field equations of GR state that only the presence of matter (or an electromagnetic field) is capable of distorting space-time in a closed system, and in my analysis I have implicitly assumed a massless, infinitesimally thin disk rotating at constant angular velocity.

However, it is possible to force a separation of four dimensional space-time into three dimensional space plus one dimensional time. Consider two points on the disk A and B separated by a small distance dx^i (i = 1, 2, 3). A measuring rod attached at A will, in the limit of very small separation, be practically at rest with respect to B. A procedure for extracting the spatial part of the metric suggests itself, and is detailed in M⌀ller [8]. It should be noted, however, that such a splitting is strictly valid only locally. M⌀ller gives the spatial part of $g_{\mu\nu}$ as,

$$\gamma_{ij} = g_{ij} + \gamma_i \gamma_j \qquad \text{where} \qquad \gamma_i \equiv \frac{g_{0i}}{\sqrt{g_{00}}}$$

Plugging the above metric into this formula yields the following spatial metric,

$$\gamma_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\rho^2}{1-\dfrac{\rho^2\omega^2}{c^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \gamma^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1-\dfrac{\rho^2\omega^2}{c^2}}{\rho^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.17}$$

and Christoffel symbols,

$$\Gamma^1_{22} = \frac{-\rho}{\left(1-\dfrac{\rho^2\omega^2}{c^2}\right)^2} \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{\rho\left(1-\dfrac{\rho^2\omega^2}{c^2}\right)} \tag{4.18}$$

with all other Γ 's zero. Thus the geodesic equations are:

$$\ddot{z} = 0 \tag{4.19}$$

$$\ddot{\rho} - \frac{\rho}{\left(1-\dfrac{\rho^2\omega^2}{c^2}\right)^2}\dot{\phi}^2 = 0 \tag{4.20}$$

$$\ddot{\phi} - \frac{1}{\rho\left(1-\dfrac{\rho^2\omega^2}{c^2}\right)}\dot{\rho}\dot{\phi} = 0 \tag{4.21}$$

At this point, one can follow M∅ller and calculate sum of the angles inside a triangle defined by such geodesics, and use this result to surmise the Gaussian curvature, or one can calculate the curvature directly. I will be following the latter approach. The formula for the curvature scalar is

$$R = g^{bd} R_{bd}, \tag{4.22}$$

where

$$R_{bd} = R^a_{bad} = \partial_a \Gamma^a_{bd} - \partial_d \Gamma^a_{ba} + \Gamma^e_{bd}\Gamma^a_{ea} - \Gamma^e_{ba}\Gamma^a_{ed} \tag{4.23}$$

$$R = \frac{6\frac{\omega^2}{c^2}}{\left(1 - \frac{\rho^2\omega^2}{c^2}\right)^2} \tag{4.24}$$

Notice that the curvature is always positive, as is the case on a sphere. This is in keeping with the previously noted conclusion that the circumference is less than $2\pi R$, by a factor that depends on the location of the observer.

5. Synchronicity on the Rim of the Disk

The Sagnac effect, and indeed all of the paradoxes described in this paper, can be explained by the impossibility of synchronizing the clocks on the disk such that all observers will agree that they are in fact synchronized. In the introduction, one of the central paradoxes is the non-transitivity of any synchronization procedure on the rim of the disk. Let us quantify this discrepancy.

Imagine two events A and B, simultaneous according to a co-moving frame K, that occur on the rim of the disk and are separated by a distance dx. Consider what happens when we transform to an adjacent reference frame, K':

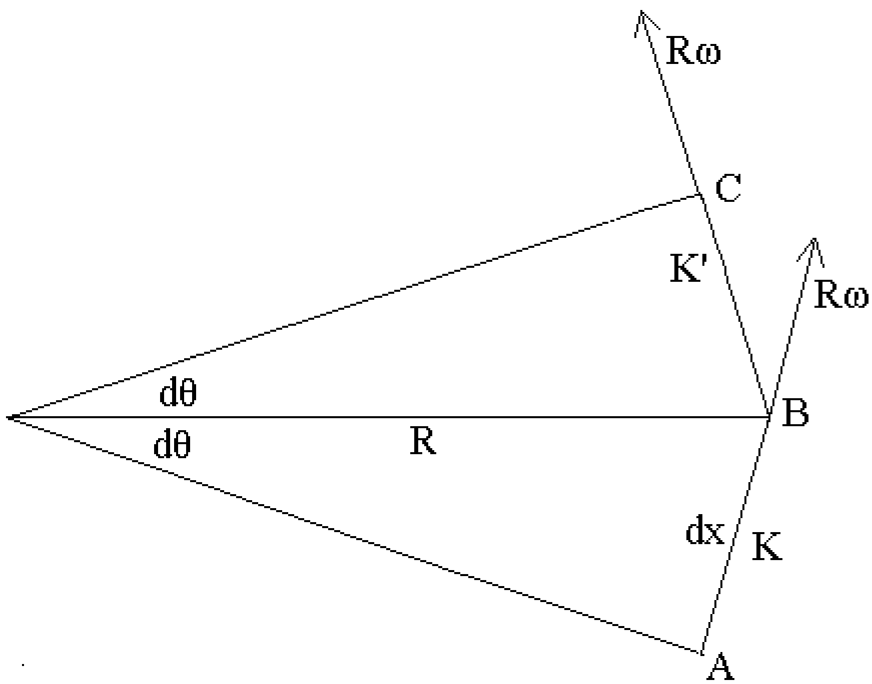

Figure 2

The time separating A and B according to observers in K' can be found via the Lorentz transformations:

$$dt' = \gamma\left(dt + \frac{vdx}{c^2}\right) = \gamma\,\frac{vRd\theta}{c^2} \quad . \tag{5.1}$$

We next consider the time difference between B and C according to a third infinitesimal frame, and continue this procedure from frame to adjacent frame all around the edge of the disk. Integrating the velocity from 0 to 2π shows that there exists a time difference that results from traversing the disk

$$\Delta t' = \gamma\,\frac{2\pi Rv}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2\pi\beta^2}{\omega\sqrt{1-\beta^2}} \quad , \tag{5.2}$$

which is precisely the time difference Sagnac effect experiments detect (as a phase shift between co- and counter-rotating laser beams). [4] [5] [6]

We see that the desynchronization is basically a different manifestation of the twin paradox. Like the twin paradox, changing between different inertial frames is the source of the discrepancy, although in this case instead of one big change, there are an infinite number of infinitesimal transformations. Furthermore, the time difference between a clock that has made one revolution around the disk and one that has remained at rest relative to the disk is an *objective* effect. That is, unlike disagreements between different inertial frames over matters of simultaneity, all observers will agree that the desynchronization given by (5.2) has taken place.

6. Sources of Curvature

In the absence of any matter or other sources of gravitational curvature a question naturally arises in the course of such curvature calculations: where does the curvature (4.22) come from? Is it real? In the 3+1 dimensions that we are accustomed to thinking in, matters are more confused, but in 4 dimensional space-time, there is little ambiguity. Much light can be shed on the issue with the aid of a simple space-time diagram. Since the disk is assumed to be infinitely thin, and nothing particularly interesting happens in the z direction, we can suppress the third dimension, and treat the system as two dimensional.

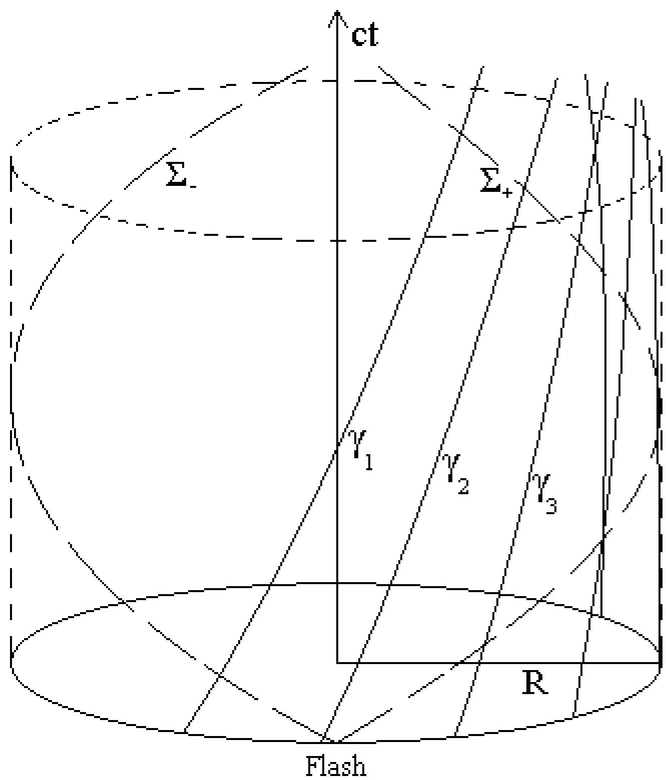

Figure 4

As shown in the above figure (which is borrowed from [5]), the world-lines of points on the rim of the disk, denoted γ_1, γ_2, γ_3, etc., are time-like helixes, as is to be expected since they are massive and must travel at less than the speed of light. The paths of the light beams are also helixes, Σ_+ and Σ_- in the diagram, which emerge from a flash and wrap around the world tube of the disk in opposite senses. This much is fairly obvious and undisputed in the literature.

The problems become evident when attempting to define a locus of simultaneity around the rim of the disk. Using the ordinary relativistic definition of "simultaneous," events simultaneous to the flash will be orthogonal to the worldlines of the observer in question (in this case, an observer riding along with the electromagnetic source that produced the flash). Thus, the integration carried out in section 5 is seen to measure an *open* spacelike curve in space-time:

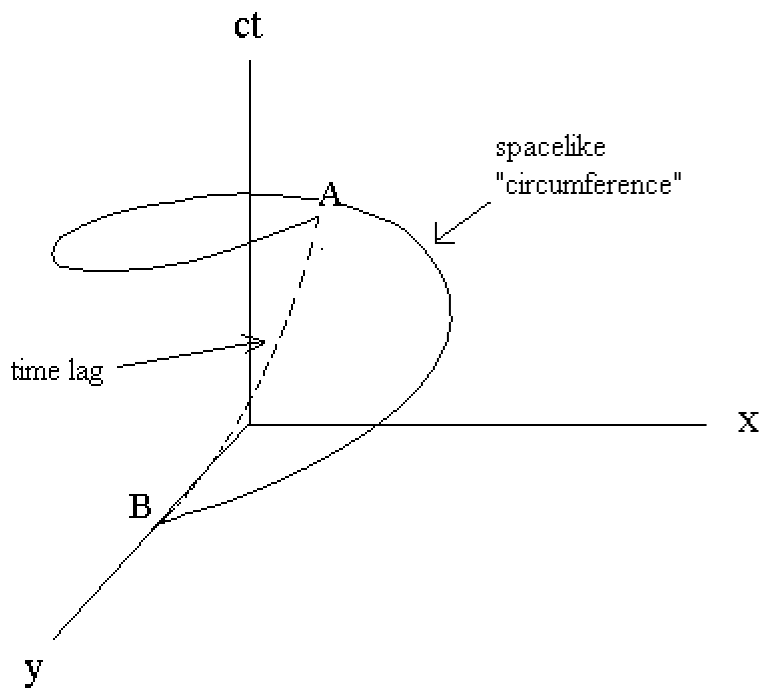

Figure 4

From figure 4, it is immediately apparent that the end of a measuring tape laid out along the periphery of the disk will not meet up with its other end at the same point in time. The two ends will be separated by a timelike path whose length is given by equation (5.2). When $\omega = 0$, the worldlines of the points on the edge of the disk are straight lines, and the locus of events simultaneous to the flash is a closed spacelike curve. But as soon as the disk is set into rotation the spacelike path changes its topology and becomes an open curve, so that the definition of simultaneity becomes a matter of convention; it depends on where one starts the integration carried out in section 5. Notice also that, combining figures 3 and 4, the angle that the light beams, Σ_+ and Σ_- make with the timelike γ 's (in fig 3) is equal to the angle between the beams and the “circumference”(in fig 4); that is, the speed of light as measured in the tangential inertial frames is simply c.

While this is all rather clear in four dimensional space-time, it is less obvious what these results mean for an experimenter on the disk. Consider an experimenter equipped with an infinite number of small measuring rods and two identical synchronized clocks. As he transverses the disk at a non-relativistic rate (relative to the rim of the disk), he lays down the measuring rods and carries one clock with him. Upon reaching his starting destination, he will conclude that the length of the rods he has laid down is the length of the circumference, namely $2\pi R\sqrt{\frac{1-\frac{v^2}{c^2}}{1+\frac{v^2}{c^2}}}$. However, he will also note that his clock lags behind the stationary clock by an amount given by (5.2). We therefore see that the curvature obtained by the methods suggested in M⌀ller are a result of artificially “forcing” the endpoints A and B in figure 4 together.

7. Conclusion

Like most relativistic paradoxes, the Ehrenfest paradox arises due to ambiguities in defining simultaneity. It is clear that most of the physicists who have previously considered the rotating disk implicitly assumed that the circumference of the disk is a well-defined geometric entity. However, by contemplating rather simple Minkowski diagrams, one comes to appreciate that a self-consistent, natural definition of simultaneity is not possible for a rapidly rotating frame. One can force an extended splitting of space-time, but the results will not necessarily coincide with any experimentally observable feature of the system (indeed, this is how the curvature calculated in section 4 appeared).

The best way to view the paradoxes of the rotating disk is as a variant on the twin paradox. It is in the changing from inertial frame to inertial frame that time is “lost.” In the words of Rizzi and Tartaglia [5],

> “…a rotating disk does not admit a well defined `proper frame’; rather, it should be regarded as a class of an infinite number of local proper frames, considered in different points at different times, and glued together according to some convention.”

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